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Convergence of a Class of Interpolatory Splines

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1. In the literature on spline interpolation, the interpolatory conditions are generally imposed at the given mesh-points (joints). In a recent paper, Subotin [*5*] has considered the existence and convergence of even degree splines with equidistant mesh-points which interpolate to given data at the mid-points of the mesh intervals. Some of these results have been treated earlier by Schoenberg [*3*] from a different point of view. The object of the present paper is to investigate the convergence properties of periodic cubic splines which interpolate to a given function at one or more inner points of the given mesh intervals. It is easy to see that if the number of interpolatory conditions in each mesh interval is more than one (as in our Theorem 2), the deficiency of the spline (see [*1*], p. 7) increases. On the other hand, the increase of deficiency of the spline (which is equivalent to the decrease of differentiability at the joints) is compensated by the increase in smoothness at the points of interpolation (in fact, we have three non-trivial continuous derivatives at the interpolation points).

Error bounds for interpolatory splines have been considered under various assumptions on the approximand by Birkhoff and de Boor [*2*] (see also [*6*], [*4*]). Here we obtain error bounds for cubic splines which interpolate at one point in each mesh interval (Section 2), and at two points in each mesh interval (Section 3). For the sake of simplicity, we consider only equidistant joints in Section 2. In Section 3, this restriction is not needed.

2. Let

$$0 = x_0 < x_1 < \ldots < x_n = 1 \tag{2.1}$$

be any subdivision of $[0,1]$. Set $h_i = x_i - x_{i-1}$, and for a given λ, $0 < \lambda \leqslant 1$, denote $t_i = x_{i-1} + \lambda h_i$, $1 \leqslant i \leqslant n$. For any given n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of real numbers, there exists a unique 1-periodic cubic spline $\phi(x) \in C^2[0,1]$ with joints $x_0, x_1, \ldots, x_n$ such that $\phi(t_i) = \alpha_i$, $1 \leqslant i \leqslant n$. The existence and uniqueness of such a spline can be proved by the methods of Ahlberg, Nilson and Walsh [*1*].

If the α_i's are the values of a 1-periodic function $f(x)$, we can formulate the following convergence theorem for equidistant joints.

THEOREM 1. *Let $f(x) \in C^2[0,1]$ be 1-periodic, and let $\phi(x) \in C^2[0,1]$ be the 1-periodic cubic spline with joints $x_i = i/n$, satisfying*

$$\phi\left(\frac{i+\lambda-1}{n}\right) = f\left(\frac{i+\lambda-1}{n}\right), \qquad i = 1, 2, \ldots, n, \tag{2.2}$$

where $0 \leqslant \lambda \leqslant \frac{1}{3}$ or $\frac{2}{3} \leqslant \lambda \leqslant 1$. If $\omega_2(\delta)$ is the modulus of continuity of $f''(x)$, then we have

$$\max_x |\phi''(x) - f''(x)| \leqslant 15\omega_2\left(\frac{1}{n}\right). \tag{2.3}$$

Remark. An analogous result where interpolation takes place between the joints rather than at the joints is due to Subotin [5]. Similar results from a different point of view go back much earlier and are due to Schoenberg [3].

An immediate consequence of (2.3) is the following:

COROLLARY. *Under the conditions of Theorem 1, we have for $r = 0, 1$,*

$$\max_x |\phi^{(r)}(x) - f^{(r)}(x)| \leqslant 15 n^{r-2}\, \omega_2\left(\frac{1}{n}\right). \tag{2.4}$$

Proof. It is easy to verify that for any cubic polynomial $p(x)$ in an interval $[a, a+h]$, and for any λ, $0 \leqslant \lambda \leqslant 1$, the following identities hold:

$$p(a) + \lambda h p'(a) + \frac{(\lambda h)^2}{2}\left(1 - \frac{\lambda}{3}\right) p''(a) + \frac{\lambda^3 h^2}{6} p''(a+h) - p(a+\lambda h) \equiv 0. \tag{2.5}$$

$$p(a+h) - (1-\lambda) h p'(a+h) + \frac{(1-\lambda)^2 h^2}{2} \cdot \frac{2+\lambda}{3} p''(a+h) + \frac{(1-\lambda)^3 h^2}{6} p''(a) - p(a+\lambda h) \equiv 0. \tag{2.6}$$

Setting

$$M_i = \phi''\left(\frac{i}{n}\right), \qquad \alpha_i = f\left(\frac{i+\lambda-1}{n}\right), \qquad \phi_i = \phi\left(\frac{i}{n}\right),$$

$$\phi_i' = \phi'\left(\frac{i}{n}\right), \qquad 0 \leqslant i \leqslant n,$$

we have on using (2.5) with

$$a = \frac{i-1}{n}, \qquad h = \frac{1}{n},$$

and (2.6) with

$$a = \frac{i-2}{n}, \qquad h = \frac{1}{n};$$

$$\phi_{i-1} = -\frac{\lambda^3}{6n^2} M_i - \frac{\lambda}{n}\phi'_{i-1} + \frac{\lambda^2}{2n^2}\left(\frac{\lambda}{3} - 1\right) M_{i-1} + \alpha_i$$
$$= -\frac{(1-\lambda)^3}{6n^2} M_{i-2} + \frac{(1-\lambda)}{n}\phi'_{i-1} - \frac{(1-\lambda)^2}{2n^2}\left(\frac{2+\lambda}{3}\right) M_{i-1} + \alpha_{i-1};$$

so that for $i = 1, 2, \ldots, n$ we have

$$n\phi'_{i-1} = \frac{(1-\lambda)^3}{6} M_{i-2} + \frac{2\lambda^3 - 3\lambda^2 - 3\lambda + 2}{6} M_{i-1} - \frac{\lambda^3}{6} M_i + (\alpha_i - \alpha_{i-1}) n^2. \quad (2.7)$$

Since $\phi''(x)$ is linear between joints, we clearly have

$$n(\phi_k' - \phi'_{k-1}) = \tfrac{1}{2}(M_k + M_{k-1}).$$

Hence (2.7), with $i = k$ and $i = k+1$, yields the following four-term relations for the M_i's:

$$\frac{(1-\lambda)^3}{6} M_{k-2} + \left(\frac{2}{3} + \frac{\lambda^3 - 2\lambda^2}{2}\right) M_{k-1} + \left(\frac{1}{6} + \frac{\lambda(1+\lambda-\lambda^2)}{2}\right) M_k + \frac{\lambda^3}{6} M_{k+1}$$
$$= 2[t_{k-1}, t_k, t_{k+1}; f], \qquad k = 1, 2, \ldots, n, \quad (2.8)$$

where $[a,b,c;f]$ denotes the usual divided difference of f at a, b, c,

$$t_i = \frac{i + \lambda - 1}{n}, \quad \text{and} \quad M_i = M_{n+i}$$

for all i.

Setting

$$f_k'' = f''\left(\frac{k}{n}\right) \quad \text{and} \quad A_k = M_k - f_k'',$$

we have from (2.8) after some simplification, for $k = 1, 2, \ldots, n$

$$\frac{(1-\lambda)^3}{6} A_{k-2} + \left(\frac{2}{3} + \frac{\lambda^3 - 2\lambda^2}{2}\right) A_{k-1} + \left(\frac{1}{6} + \frac{\lambda(1+\lambda-\lambda^2)}{2}\right) A_k + \frac{\lambda^3}{6} A_{k+1}$$
$$= f''(\eta_k) - f_k'' + \frac{(1-\lambda)^3}{6}(f_k'' - f''_{k-2})$$
$$+ \left(\frac{2}{3} + \frac{\lambda^3 - 2\lambda^2}{2}\right)(f_k'' - f''_{k-1}) + \frac{\lambda^3}{6}(f_k'' - f''_{k+1}) \quad (2.9)$$

where $t_{k-1} < \eta_k < t_{kH}$.

If $\lambda \to 1-$ or $\lambda \to 0+$, the above system of equations reduces to the system (3.1) of ([*4*] p. 761). Thus, for λ sufficiently close to 1 or for λ sufficiently small, in particular, for $\frac{2}{3} \leqslant \lambda \leqslant 1$, respectively $0 \leqslant \lambda \leqslant \frac{1}{3}$, the coefficient of A_k is larger than the sum of the coefficients of A_{k-2}, A_{k-1} and A_{k+1}, respectively the coefficient of A_{k-1} is larger than the sum of the coefficients of A_{k-2}, A_k, A_{k+1}

in (2.9). Hence by a reasoning used already in [4], we have for $0 \leqslant \lambda \leqslant \frac{1}{3}$ or $\frac{2}{3} \leqslant \lambda \leqslant 1$,

$$\max_k |A_k| \leqslant 14\omega_2\left(\frac{1}{n}\right). \tag{2.10}$$

Since $\phi''(x)$ is linear between joints, it follows easily from (2.10) that

$$|\phi''(x) - f''(x)| \leqslant 15\omega_2\left(\frac{1}{n}\right)$$

uniformly in $[0,1]$. This completes the proof of Theorem 1 for $\lambda \geqslant \frac{2}{3}$.

The corollary follows easily on observing that $\phi'(x) - f'(x)$ vanishes for at least one η_i in the interior of the interval

$$\left(\frac{i+\lambda-1}{n}, \frac{i+\lambda}{n}\right), \qquad i = 1, 2, \ldots, n$$

by Rolle's theorem. If

$$x \in \left[\frac{i+\lambda-1}{n}, \frac{i+\lambda}{n}\right], \quad \text{then} \quad |x - \eta_i| < \frac{1}{n},$$

so that by (2.3),

$$|\phi'(x) - f'(x)| \leqslant \frac{15}{n}\omega_2\left(\frac{1}{n}\right).$$

A further integration yields (2.4) for $r = 0$.

3. Deficient Cubic Spline. Since there is no a priori reason for having only one point of interpolation between two successive joints, it is natural to inquire into the behaviour of cubic splines interpolating in two or more points in each subinterval formed by the joints. This additional constraint naturally increases the deficiency of the spline curve at the joints. We shall not assume here that the joints are equispaced; however, we shall restrict ourselves to the case where the points of interpolation follow the same pattern in each subinterval. More precisely, if (2.1) is a given subdivision of $[0,1]$, l, m $(0 \leqslant m < l \leqslant 1)$ are given real numbers with $\frac{1}{2} < l + m < \frac{3}{2}$, $(\alpha_1, \alpha_2, \ldots \alpha_n)$ and $(\beta_1, \beta_2, \ldots, \beta_n)$ are given n-tuples of reals, then there exists a unique 1-periodic cubic spline $\phi(x) \in C^1[0,1]$ with joints (2.1), such that

$$\phi(\xi_i) = \alpha_i, \qquad \phi(\eta_i) = \beta_i, \qquad i = 1, 2, \ldots, n,$$

where

$$\xi_i = mx_i + (1-m)x_{i-1}, \qquad \eta_i = lx_i + (1-l)x_{i-1}.$$

The proof of this assertion can be carried out along the usual lines [6]. Our object here is to prove the following convergence theorem.

THEOREM 2. *Let* $f(x) \in C^1[0,1]$ *be* 1-*periodic and let* $\phi(x) \in C^1[0,1]$ *be the* 1-*periodic cubic spline satisfying*

$$\phi(\xi_i) = f(\xi_i) \equiv \alpha_i, \qquad \phi(\eta_i) = f(\eta_i) \equiv \beta_i, \qquad i = 1, 2, \ldots, n. \tag{3.1}$$

If $\omega_1(\delta)$ *is the modulus of continuity of* $f'(x)$, *we have for* $r = 0, 1$

$$\max_x |\phi^{(r)}(x) - f^{(r)}(x)| \leqslant K(l,m).\omega_1(\Delta).\Delta^{1-r}, \tag{3.2}$$

where $\Delta = \max_i h_i$ *and* $K(l,m)$ *depends on* l *and* m *only.*

For the proof of this theorem we shall need the following

LEMMA. *If* $P(x)$ *is any cubic polynomial in* $[a,b]$, $\xi = mb + (1-m)a$, $\eta = lb + (1-l)a$, $0 \leqslant m < l \leqslant 1$, *then the following identity is valid:*

$$AP(a) = BP(\xi) + CP(\eta) + D(b-a)P'(b) + E(b-a)P'(a) \tag{3.3}$$

where

$$\left.\begin{aligned} &A = l^3 - m^3 - \tfrac{3}{2}(l^2 - m^2), \qquad B = l^2(l - \tfrac{3}{2}),\\ &C = m^2(\tfrac{3}{2} - m), \qquad D = \frac{l^2 m^2(-l+m)}{2},\\ &E = lm(m-l)(2l + 2m - lm - 3). \end{aligned}\right\} \tag{3.4}$$

This identity is easy to verify.

Proof of Theorem 2. For the sake of brevity, set $N_i = \phi'(x_i)$, $\phi_i = \phi(x_i)$, $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$. Then using (3.3) first with $a = x_i$, $b = x_{i+1}$, $\xi = \xi_{i+1}$, $\eta = \eta_{i+1}$, next with $a = x_i$, $b = x_{i-1}$, $1-l$ for l, and $1-m$ for m, and eliminating ϕ_i from the two equations so obtained, we have for $i = 1, 2, \ldots, n$,

$$\begin{aligned} l^2(l-\tfrac{3}{2})\,\alpha_{i+1} &+ m^2(\tfrac{3}{2}-m)\,\beta_{i+1} + \tfrac{1}{2}l^2 m^2(m-l)\,h_{i+1}N_{i+1}\\ &+ \tfrac{1}{2}lm(m-l)(1-(2-l)(2-m))\,h_{i+1}N_i - (1-l)^2(l+\tfrac{1}{2})\,\alpha_i\\ &- (1-m)^2(\tfrac{1}{2}+m)\,\beta_i - \tfrac{1}{2}(1-l)^2(1-m)^2(l-m)\,h_i N_{i-1}\\ &- \tfrac{1}{2}(1-l)(1-m)(l-m)(1-(1+l)(1+m))\,h_i N_i = 0, \end{aligned}$$

where $N_{i+n} = N_i$ for all i.

A further simplification yields the following system of three-term relations:

$$\begin{aligned} -p_1 N_{i-1} h_i &+ (p_2 h_i + p_3 h_{i+1}) N_i - p_4 h_{i+1} N_{i+1}\\ &= \frac{2}{l-m}\big[(l+\tfrac{1}{2})(1-l)^2\,\alpha_i - (m+\tfrac{1}{2})(1-m)^2\,\beta_i\\ &\qquad + (\tfrac{3}{2}-l)\,l^2\,\alpha_{i+1} - (\tfrac{3}{2}-m)\,m^2\,\beta_{i+1}\big], \qquad i = 1, 2, \ldots, n, \end{aligned} \tag{3.5}$$

where

$$\left. \begin{array}{ll} p_1 = (1-m)^2(1-l)^2, & p_2 = (1-m)(1-l)\{(1+l)(1+m)-1\}, \\ p_3 = lm\{(2-l)(2-m)-1\}, & p_4 = l^2 m^2. \end{array} \right\} \quad (3.6)$$

Since the spline ϕ interpolates to f at the points ξ_i, η_i $(i = 1,2,\ldots,n)$, we have

$$\begin{aligned} \beta_i = f(\eta_i) &= f(\xi_i) + (l-m) h_i f'(\rho_i) \\ &= \alpha_i + (l-m) h_i f'(\rho_i), \qquad \xi_i < \rho_i < \eta_i. \end{aligned} \quad (3.7)$$

Using (3.7) in (3.5) to replace β_{i+1} and α_i, the right side of (3.5) becomes, after using the mean-value theorem and the Darboux property pf a derivative:

$$\begin{aligned} &2(l^2 + lm + m^2 - \tfrac{3}{2}l - \tfrac{3}{2}m)(\beta_i - \alpha_{i+1}) \\ &\quad - 2(l + \tfrac{1}{2})(1-l)^2 h_i f'(\rho_i) - 2(\tfrac{3}{2} - m) m^2 h_{i+1} f'(\rho_{i+1}) \\ &\qquad = q_1((1-l) h_i + m h_{i+1}) f'(\sigma_i) - [q_2 h_i + q_3 h_{i+1}] f'(\tau_{i+1}), \end{aligned} \quad (3.8)$$

where

$$\begin{array}{lll} q_1 = 3l + 3m - 2l^2 - 2lm - 2m^2, & q_2 = 2(l + \tfrac{1}{2})(1-l)^2, & \\ q_3 = 2(\tfrac{3}{2} - m) m^2, & x_{i-1} < \tau_{i+1} < x_{i+1}, & x_{i-1} < \sigma_i < x_{i+1}. \end{array}$$

Now, setting $B_i = N_i - f_i'$ $(1 \leqslant i \leqslant n)$, and using (3.8), we have from (3.5) the following system of equations:

$$\begin{aligned} &-p_1 h_i B_{i-1} + (p_2 h_i + p_3 h_{i+1}) B_i - p_4 h_{i+1} B_{i+1} \\ &\quad = q_1((1-l) h_i + m h_{i+1})(f'(\sigma_i) - f_i') \\ &\qquad - (q_2 h_i + q_3 h_{i+1})(f'(\tau_{i+1}) - f_i') \\ &\qquad + p_1 h_i (f'_{i-1} - f_i') + p_4 h_{i+1}(f'_{i+1} - f_i'), \qquad i = 1, 2, \ldots, n. \end{aligned} \quad (3.9)$$

Since $\frac{1}{2} < l + m < \frac{3}{2}$, $p_2 - p_1 = (1-l)(1-m)(2l + 2m - 1) > 0$ and $p_3 - p_4 = lm(3 - 2l - 2m) > 0$, so that the method of [*4*] can be used to find an upper bound for $\max_i |B_i|$. For, if $\max_i |B_i| = |B_j|$, then

$$\begin{aligned} &\{(p_2 - p_1) h_j + (p_3 - p_4) h_{j+1}\} |B_j| \\ &\qquad \leqslant \{(q_1(1-l) + q_2 + p_1) h_j + (m q_1 + q_3 + p_4) h_{j+1}\} \omega_1(\Delta). \end{aligned}$$

Since $p_2 - p_1$ and $p_3 - p_4$ are positive numbers depending only on l and m, it follows that

$$\max_i |B_i| \leqslant K_1(l,m)\, \omega_1(\Delta), \quad (3.10)$$

where $K_1(l,m)$ is independent of the choice of the joints. This proves that as $\Delta \to 0$, the difference $\phi_i' - f_i'$ tends to zero uniformly at all joints.

It remains to prove that $\phi'(x) - f'(x)$ also approaches zero as Δ tends to zero. Now, for $x_{i-1} \leqslant x \leqslant x_i$, $\phi(x) = \lambda_i(x) + \psi_i(x)$, where $y = \lambda_i(x)$ is the straight line through the points (ξ_i, α_i) and (η_i, β_i). Hence

$$\psi_i(x) = \gamma_i (x - \xi_i)(x - \eta_i)(x - \zeta_i),$$

with a suitable γ_i and ζ_i, so that

$$N_{i-1} = \lambda_i'(x) + \gamma_i[(2x_{i-1} - \xi_i - \eta_i)(x_{i-1} - \zeta_i) + (x_{i-1} - \xi_i)(x_{i-1} - \eta_i)]$$
$$= \lambda_i'(x) + \gamma_i[(\xi_i - x_{i-1})(l+m)h_i + lmh_i^2].$$

Similarly,

$$N_i = \lambda_i'(x) + \gamma_i[(2-l-m)h_i(x_i - \zeta_i) + (1-l)(1-m)h_i^2].$$

Hence,

$$(2-l-m)N_{i-1} + (l+m)N_i - 2\lambda_i'(x)$$
$$= \gamma_i[(2-l-m)(l+m) + lm(2-l-m) + (1-l)(1-m)(l+m)]h_i^2$$
$$= \gamma_i h_i^2[3(l+m) - 2(l^2 + lm + m^2)]. \tag{3.11}$$

Also, from the definition of the N_i's, it follows that

$$\phi'(x) = \Lambda_i(x) + 3\gamma_i(x - x_i)(x - x_{i-1}),$$

with

$$\Lambda_i(x) = N_{i-1}\frac{x_i - x}{h_i} + N_i\frac{x - x_{i-1}}{h_i}.$$

Thus,

$$|\phi'(x) - \Lambda_i(x)| \leqslant 3|\gamma_i||(x - x_i)(x - x_{i-1})| \leqslant \tfrac{3}{4}|\gamma_i|h_i^2$$

so that using (3.11), we have

$$|\phi'(x) - \Lambda_i(x)| \leqslant \frac{3}{4}\frac{|(2-l-m)N_{i-1} + (l+m)N_i - 2\lambda_i'(x)|}{|3(l+m) - 2(l^2 + lm + m^2)|}. \tag{3.12}$$

Since

$$\lambda_i'(x) = \frac{\beta_i - \alpha_i}{\eta_i - \xi_i} = f'(\theta_i), \qquad \xi_i < \theta_i < \eta_i,$$

the numerator on the right side of (3.12) is

$$\leqslant (2-l-m)|B_{i-1}| + (l+m)|B_i|$$
$$+ (2-l-m)|f_{i-1}' - f''(\theta_i)| + (l+m)|f_i' - f'(\theta_i)|$$
$$\leqslant 2(K_1(l,m) + 1)\,\omega_1(\Delta).$$

Therefore, from (3.12), we obtain

$$|\phi'(x) - \Lambda_i(x)| \leqslant K_2(l,m)\,\omega_1(\Delta).$$

Denoting by $y = \Lambda_i^*(x)$ the straight line through the points (x_{i-1}, f_{i-1}') and (x_i, f_i'), and observing that $|\Lambda_i(x) - \Lambda_i^*(x)| \leqslant \max_i |B_i|$ for all x in $[x_{i-1}, x_i]$, we have

$$|\phi'(x) - f'(x)| \leqslant |\phi'(x) - \Lambda_i(x)| + |\Lambda_i(x) - \Lambda_i^*(x)| + |\Lambda_i^*(x) - f'(x)|$$
$$\leqslant K(l,m)\,\omega_1(\Delta).$$

4. The Special Case $l+m=1$. When the points ξ_i, η_i of Theorem 2 are symmetrically situated in the interval (x_{i-1}, x_i), which corresponds to the condition $l=1-m$, then the system of equations (3.5) becomes considerably simpler and a numerical estimate for the constant $K(l,m)$ of (3.2) can be easily obtained. In fact, in this case $K_1(l,m)$ of (3.10) can be replaced by the constant $\frac{21}{4}$ and some further computation shows that (3.2) holds with $K(l,m)$ replaced by the constant 16.

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